Asymptotic Solution of a Thick Spherical Shell with Circular Holes

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A singular perturbation analysis is developed for a spherical shell, containing two diametrically opposite holes, subjected to an axisymmetric external pressure. The asymptotic formulation decomposes the shell into three subregions: an interior region, a wide boundary layer, and a narrow boundary layer. The subregional solutions are matched and a uniformly valid solution is obtained. Stress concentration factors, radial normal and shearing stresses, and displacements are presented for the case of a uniform external load p acting on a spherical shell, containing two diametrically opposite holes, each of which is subtended by a half angle of one-tenth of a radian $\varphi_o = 0.1$. It is shown that the narrow boundary-layer region is of the order of the shell thickness, while the wide boundary-layer effects may be neglected at distances greater than 2 $(Rh)^{1/2}$ from the hole.

Nomenclature

E	= Young's modulus
h	= shell thickness
K	= stress concentration factor, $K = \sigma_{\theta\theta}/\sigma_{\theta\theta}$ (full sphere)
K_o, K_{+1}, K_{-1}	= stress concentration factor at median, outer and inner surfaces, respectively
$P(\varphi)$	= nondimensional external pressure
$p(\varphi)$	= pressure coefficient, $p(\varphi) = P(\varphi)/\epsilon^2$
R	= mean shell radius
r^*	= radial coordinate
r	= nondimensional radial coordinate, $r = r^*/R$
$u_r^*, u_\theta^*, u_\varphi^*$	= radial, circumferential, and meridional displacements
u_r, u_θ, u_φ	= nondimensional displacements, $u_r = u_r^*/R$
y	= narrow boundary layer stretched meridional coordinate, $y = (\varphi - \varphi_o)/\epsilon^2$
z	= stretched radial coordinate measured from median surface, $z = (r - 1)/\epsilon^2$
ϵ	= thickness to radius parameter, $\epsilon = (h/2R)^{1/2}$
θ	= angle of longitude (circumferential coordinate)
ν	= Poisson's ratio
$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\varphi\varphi}$	= nondimensional radial, circumferential, and meridional normal stresses
$\sigma_{r\theta}, \sigma_{r\varphi}, \sigma_{\theta\varphi}$	= nondimensional shearing stresses
φ	= meridional coordinate
φ_o	= half-angle subtended by cutout
ψ	= stress function

Introduction

MANY problems of mathematical physics have been successfully investigated by use of perturbation or asymptotic methods.[1-3] Their solutions are obtained by constructing an asymptotic expansion with respect to a small (or large) parameter. In employing these procedures two types of problems are encountered, namely, the regular and singular perturbation problems. The solutions of the regular perturbation problems are uniformly valid throughout the regions of interest, while for the singular perturbations more than one solution exists, each valid in a subregion. These singular perturbation solutions must therefore be combined so as to obtain a uniformly valid solution.

Singular perturbation problems arise as a consequence of a rapid transition which occurs somewhere in the region of interest. This subregion of rapid transition often occurs at a boundary, and in such cases is termed a boundary layer or edge effect. Usually, though not always, such problems exhibit a reduction in the order of the governing equations as the small (large) parameter tends to zero (infinity), thus reducing the number of boundary conditions which may be satisfied. A classical example of such a phenomenon is the boundary layer effect so successfully analyzed by Prandtl.[4]

In the field of solid mechanics, singular perturbation theory has been successfully applied to the study of edge effects. Friedrichs,[5,6] Friedrichs and Dressler,[7] and Reiss[8] use asymptotic methods to justify the use of the Kirchhoff boundary conditions for lower order theories, in the bending of plates. Friedrichs and Stoker[9] investigated the large deflections of plates by using asymptotic methods. Reiss and Locke,[10] extending the work of Friedrichs and Dressler, investigated the three-dimensional problem for thin plates and showed that the zeroth- and first-order interior solutions coincided with the generalized plane stress formulation. Reiss[11] investigated the stress concentration due to extension of an infinite plate with a hole, while Burniston[12] considered the same problem under flexural loads. Shell theory edge effects have also been treated as asymptotic phenomena.[13-15] Furthermore, asymptotic expansions have been used to construct approximate two-dimensional shell theories from the three-dimensional elasticity formulation.[16,17]

The present investigation concerns itself with a spherical shell containing two diametrically opposite holes. The shell is subjected to an axisymmetric external pressure. The asymptotic formulation decomposes the shell into three subregions: 1) an interior region (away from the holes), 2) a wide boundary layer (closer to the holes), and 3) a narrow boundary layer (adjacent to the holes) each of which is characterized by different order of derivatives of the dependent variables. The differences of these rates are incorporated into the problem by appropriately stretching the meridional coordinate. The forms of the governing equations in the interior and wide boundary layer are essentially

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those of the membrane and bending shell theories, respectively. On the other hand, the lower-order forms obtained for the governing equations in the narrow boundary layer coincide with the plane strain biharmonic forms. The subregional solutions are matched and a solution which is uniformly valid throughout the shell is obtained.

The biharmonic equations are solved by employing a series composed of Fadle-Papkovitch functions.[18,19] Several approximate methods for determining the coefficients of the nonorthogonal series have been proposed and are discussed by Levine and Klosner.[20] Here a least-square point matching technique is used to obtain these coefficients.[20–24]

The authors would like to point out that the singular perturbation analysis developed here is applicable to spherical shells containing two diametrically opposite (stress-free) holes and subjected to an axisymmetric external pressure. However, with but a slight modification it can be applied to a real-life structural problem where the holes would be subjected to interaction stresses. For such a loading, even for thin shells, shell theory would be totally inadequate in predicting the very significant shearing stresses that would appear in the narrow boundary layer (which is of the order of the shell thickness) adjacent to the hole. Thus, a formal elasticity solution would have to be developed. But, such a difficult and lengthy procedure[20] can be circumvented by applying the perturbation analysis (of the elasticity boundary-value problem) which is presented here.

Formulation of the Problem

Consider an axisymmetric loaded spherical shell with circular cutouts at each pole, subtending the half-angle φ_0 (Fig. 1). The dependent functions are thus independent of θ, the angle of longitude, and $\sigma_{r\theta} = \sigma_{\rho\theta} = u_\theta = 0$.

The governing equilibrium equations expressed in spherical coordinates are

$$\frac{1}{r^2}(r^2\sigma_{rr})_{,r} + \frac{1}{r\sin\varphi}(\sin\varphi\,\sigma_{r\varphi})_{,\varphi} - \frac{1}{r}(\sigma_{\varphi\varphi} + \sigma_{\theta\theta}) = 0$$
$$\frac{1}{r^3}(r^3\sigma_{r\varphi})_{,r} + \frac{1}{r\sin\varphi}(\sin\varphi\,\sigma_{\varphi\varphi})_{,\varphi} - \frac{\cot\varphi}{r}\sigma_{\theta\theta} = 0 \tag{1}$$

where $r = r^*/R$ is the nondimensional radial coordinate, R is the mean shell radius, φ is the meridional angle, and the stresses have been nondimensionalized, i.e., for example, $\sigma_{rr} = \sigma_{rr}^*/E$.

The stress displacement relations are

$$u_{r,r} = \sigma_{rr} - \nu(\sigma_{\varphi\varphi} + \sigma_{\theta\theta})$$
$$\frac{1}{r}(u_{\varphi,\varphi} + u_r) = \sigma_{\varphi\varphi} - \nu(\sigma_{\theta\theta} + \sigma_{rr})$$
$$\frac{1}{r}(u_r + \cot\varphi\,u_\varphi) = \sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{\varphi\varphi}) \tag{2}$$
$$\frac{1}{r}u_{r,\varphi} + u_{\varphi,r} - \frac{1}{r}u_\varphi = 2(1+\nu)\sigma_{r\varphi}$$

where the displacements have been nondimensionalized with respect to the mean radius, so that $u_r = u_r^*/R$.

The four nonvanishing compatibility equations may be expressed as

$$\frac{r}{\sin\varphi}(\sin\varphi\,\sigma_{\theta\theta,r})_{,\varphi} - \sigma_{rr,\varphi} - r\cot\varphi\,\sigma_{\varphi\varphi,r} + 2\sigma_{r\varphi} - \left(\frac{\nu}{1+\nu}\right)r^2\left(\frac{1}{r}\sigma_{,\varphi}\right)_{,r} = 0$$

$$\sigma_{rr,\varphi\varphi} + (r^2\sigma_{\varphi\varphi,r})_{,r} - r\sigma_{rr,r} - 2(r\sigma_{r\varphi,\varphi})_{,r} - \left(\frac{\nu}{1+\nu}\right)[\sigma_{,\varphi\varphi} + r(r\sigma_{,r})_{,r}] = 0$$

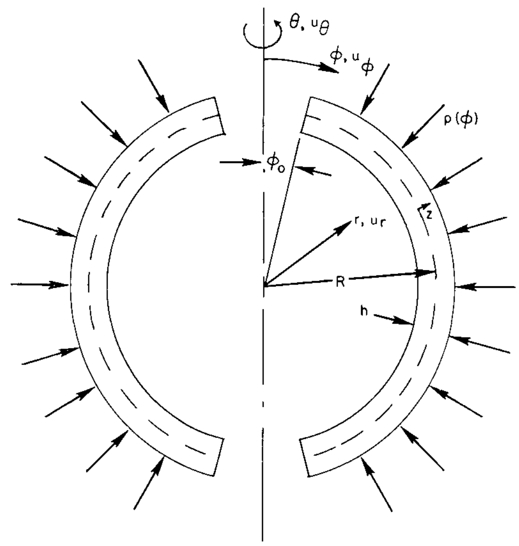

Fig. 1 Shell geometry.

$$\frac{1}{\sin\varphi}(\sin\varphi\,\sigma_{\theta\theta,\varphi})_{,\varphi} + \cot\varphi(\sigma_{\theta\theta} - \sigma_{\varphi\varphi})_{,\varphi} + r(\sigma_{\theta\theta} + \sigma_{\varphi\varphi})_{,r} + 2(\sigma_{\varphi\varphi} - \sigma_{rr})_{,\varphi} - \frac{2}{\sin\varphi}(\sin\varphi\,\sigma_{r\varphi})_{,\varphi} - \left(\frac{\nu}{1+\nu}\right)\left[\frac{1}{\sin\varphi}(\sin\varphi\,\sigma_{,\varphi})_{,\varphi} + 2r\sigma_{,r}\right] = 0 \tag{3}$$

$$(r^2\sigma_{\theta\theta,r})_{,r} - r\sigma_{rr,r} + \cot\varphi\,\sigma_{rr,\varphi} - 2\cot\varphi(r\sigma_{r\varphi})_{,r} - \left(\frac{\nu}{1+\nu}\right)[r(r\sigma_{,r})_{,r} + \cot\varphi\,\sigma_{,\varphi}] = 0$$

where the first stress invariant

$$\sigma = \sigma_{rr} + \sigma_{\varphi\varphi} + \sigma_{\theta\theta}$$

The boundary conditions at the inner and outer shell surfaces are

$$\sigma_{rr}[\varphi,1 + (h/2R)] = -P(\varphi), \sigma_{rr}[\varphi,1 - (h/2R)] = 0$$
$$\sigma_{r\varphi}[\varphi,1 + (h/2R)] = \sigma_{r\varphi}[\varphi,1 - (h/2R)] = 0 \tag{4}$$

and those at the edge of the hole are

$$\sigma_{\varphi\varphi}(\varphi_0,r) = \sigma_{r\varphi}(\varphi_0,r) = 0 \tag{5}$$

Also from symmetry at $\varphi = \pi/2$

$$\sigma_{r\varphi}[(\pi/2),r] = 0, \; u_\varphi[(\pi/2),r] = 0 \tag{6}$$

The problem may be formulated in either of two ways. The first approach makes use of the equilibrium equations (1), the stress-displacement relations (2), and the boundary conditions (4–6). The alternate approach makes use of the equilibrium equations (1), the compatibility equations (3), and the boundary conditions (4–6).

The Interior Problem

We define the parameter

$$\epsilon = (h/2R)^{1/2} \tag{7}$$

and make the assumption that each of the components of the stresses and displacements may be written as a power series in ϵ, although it should be pointed out that this assumption is not always valid.[3] Let f be the generic symbol denoting any stress or displacement component. Thus we assume that

$$f(\varphi,z,\epsilon) \equiv \sum_{n=0}^{\infty} f^{(n)}(\varphi,z)\epsilon^n \quad (8)$$

where, for convenience, $f^{(n)} = 0$ if $n < 0$, and the transformation coordinate

$$z = (r-1)/\epsilon^2 \quad (9)$$

The asymptotic forms of the equilibrium and stress displacement relations are obtained by substituting Eqs. (8) and (9) into Eqs. (1) and (2) and equating the coefficients of the same power of ϵ^n, the asymptotic forms of the equilibrium equations are

$$\sigma_{rr,z}^{(n+2)} = -2\sigma_{rr}^{(n)} - z\sigma_{rr,z}^{(n)} - \sigma_{r\varphi,\varphi}^{(n)} + \cot\varphi\sigma_{r\varphi}^{(n)} + \sigma_{\varphi\varphi}^{(n)} + \sigma_{\theta\theta}^{(n)} \quad (10a)$$

$$\sigma_{r\varphi,z}^{(n+2)} = -3\sigma_{r\varphi}^{(n)} - z\sigma_{r\varphi,z}^{(n)} - \sigma_{\varphi\varphi,\varphi}^{(n)} + \cot\varphi(\sigma_{\theta\theta}^{(n)} - \sigma_{\varphi\varphi}^{(n)}) \quad (10b)$$

while the stress displacement relations can be written as

$$u_{r,z}^{(n)} = \sigma_{rr}^{(n-2)} - \nu(\sigma_{\varphi\varphi}^{(n-2)} + \sigma_{\theta\theta}^{(n-2)}) \quad (11a)$$

$$u_{\varphi,\varphi}^{(n)} + u_r^{(n)} = [\sigma_{\varphi\varphi}^{(n)} - \nu(\sigma_{\theta\theta}^{(n)} + \sigma_{rr}^{(n)})] + z[\sigma_{\varphi\varphi}^{(n-2)} - \nu(\sigma_{\theta\theta}^{(n-2)} + \sigma_{rr}^{(n-2)})] \quad (11b)$$

$$u_r^{(n)} + \cot\varphi u_\varphi^{(n)} = [\sigma_{\theta\theta}^{(n)} - \nu(\sigma_{rr}^{(n)} + \sigma_{\varphi\varphi}^{(n)})] + z[\sigma_{\theta\theta}^{(n-2)} - \nu(\sigma_{rr}^{(n-2)} + \sigma_{\varphi\varphi}^{(n-2)})] \quad (11c)$$

$$u_{\varphi,z}^{(n)} + zu_{\varphi,z}^{(n-2)} + u_{r,\varphi}^{(n-2)} - u_\varphi^{(n-2)} = 2(1+\nu)\sigma_{r\varphi}^{(n)} + 2z(1+\nu)\sigma_{r\varphi}^{(n-2)} \quad (11d)$$

Substitution of $n = -2$ and -1 into the equilibrium equations (10) yields

$$\sigma_{rr,z}^{(0)} = \sigma_{rr,z}^{(1)} = \sigma_{r\varphi,z}^{(0)} = \sigma_{r\varphi,z}^{(1)} = 0 \quad (12)$$

and, therefore, the above stresses are only functions of φ. To satisfy the boundary conditions (4) at the inner and outer surfaces, it thus is necessary that

$$\sigma_{rr}^{(0)} = \sigma_{rr}^{(1)} = \sigma_{r\varphi}^{(0)} = \sigma_{r\varphi}^{(1)} = 0 \quad (13)$$

At most $\sigma_{rr}^{(0)}$ and $\sigma_{rr}^{(1)}$ could have been constants which would have satisfied a condition of uniform stretch or contraction through the shell's thickness. Since we are not interested in this type of loading, the radial stress conditions at the inner and outer surfaces must be at least of order ϵ^2. We therefore express the function $P(\varphi)$ of Eq. (4) as

$$P(\varphi) = p(\varphi)\epsilon^2 \quad (14)$$

although the choice of a higher order form would not have affected the final solution.

Here the boundary conditions (4) and (5) are

$$\sigma_{r\varphi}^{(n)}(\pi/2,z) = 0; \quad u_\varphi^{(n)}(\pi/2,z) = 0; \quad \sigma_{r\varphi}^{(n)}(\varphi,\pm 1) = 0;$$
$$\sigma_{rr}^{(n)}(\varphi,-1) = 0; \quad \sigma_{rr}^{(n)}(\varphi,+1) = 0, n \neq 2; \quad (15)$$
$$\sigma_{rr}^{(2)}(\varphi,+1) = -p(\varphi)$$

However, the above conditions are not sufficient to fully specify the interior problem. One additional condition is required, and that is obtained by matching with the wide boundary-layer solution.

We now proceed to obtain the forms of the displacement functions, stress displacement relations and equilibrium equations for the first two terms of the expansion. (For higher order forms in all regions see Ref. 25.)

For $n = 0$ or 1, Eqs. (11a) and (11d) yield the following restricted forms for the displacements

$$u_r^{(n)}(\varphi,z) = U_r^{(n)}(\varphi) \qquad u_\varphi^{(n)}(\varphi,z) = U_\varphi^{(n)}(\varphi) \quad (16)$$

Combining Eqs. (11b) and (11c), we obtain the stress displacement relations

$$\sigma_{\varphi\varphi}^{(n)}(\varphi,z) = \frac{1}{1-\nu^2}\left[(1+\nu)U_r^{(n)} + \frac{dU_\varphi^{(n)}}{d\varphi} + \nu\cot\varphi U_\varphi^{(n)}\right] = \Sigma_{\varphi\varphi}^{(n)}(\varphi)$$
$$(17)$$
$$\sigma_{\theta\theta}^{(n)}(\varphi,z) = \frac{1}{1-\nu^2}\left[(1+\nu)U_r^{(n)} + \nu\frac{dU_\varphi^{(n)}}{d\varphi} + \cot\varphi U_\varphi^{(n)}\right] = \Sigma_{\theta\theta}^{(n)}(\varphi)$$

The equilibrium equations and the surface boundary condition yield

$$\sigma_{r\varphi}^{(n+2)}(\varphi,z) = 0$$
$$\sigma_{rr}^{(n+2)}(\varphi,z) = \begin{cases} -\frac{1}{2}p(1+z), & n = 0 \\ 0 \quad, & n = 1 \end{cases} \quad (18)$$

and $$\Sigma_{\varphi\varphi}^{(n)} + \Sigma_{\theta\theta}^{(n)} = \begin{cases} -p/2, & n = 0 \\ 0 \quad, & n = 1 \end{cases}$$

$$[d\Sigma_{\varphi\varphi}^{(n)}]/d\varphi + \cot\varphi(\Sigma_{\varphi\varphi}^{(n)} - \Sigma_{\theta\theta}^{(n)}) = 0 \quad (19)$$

The equilibrium equations (19), in terms of the displacements, are

$$2U_r^{(n)} + \frac{dU_\varphi^{(n)}}{d\varphi} + \cot\varphi U_\varphi^{(n)} = \begin{cases} -\frac{(1-\nu)}{2}p, & n = 0 \\ 0 \quad, & n = 1 \end{cases}$$
$$(20)$$
$$(1+\nu)\frac{dU_r^{(n)}}{d\varphi} + \frac{d^2U_\varphi^{(n)}}{d\varphi^2} + \cot\varphi\frac{dU_\varphi^{(n)}}{d\varphi} - (\nu + \cot^2\varphi)U_\varphi^{(n)} = 0$$

Also, at $\varphi = \pi/2$ we have

$$U_\varphi^{(n)}(\pi/2) = 0 \quad (21)$$

Since the above set of equations and boundary conditions is homogeneous for $n = 1$ because $\Sigma_{\varphi\varphi}^{(1)}(\varphi_0) = 0$,[25] the corresponding functions vanish identically. Furthermore, since Eqs. (10) and (11) reveal that (odd) even ordered functions are only coupled by (odd) even ordered functions, all odd ordered functions vanish.

It should be pointed out that the equilibrium equations (19) and the stress displacement relations (17) for $n = 0,1$ are identical to those of the classical membrane theory. The radial stresses (18) are not contained in the membrane theory.

The Wide Boundary-Layer Problem

A. Formulation

It is obvious that the stresses of the interior problem, which are explicit functions of z, cannot satisfy the given arbitrary boundary stresses. Furthermore, it is shown that the "elasticity" boundary layer cannot be used to match the interior stresses. Thus some intermediate boundary layer must be constructed having a width that is greater than that of the narrow (elasticity) boundary layer. We thus introduce a new meridional coordinate

$$x = (\varphi - \varphi_0)/\epsilon \quad (22)$$

Thus, $\cot\varphi = \cot(\epsilon x + \varphi_o)$, and its expansion is

$$\cot\varphi = k_1 - \epsilon k_2^2 + \ldots; \quad k_1 = \cot\varphi_0, k_2 = csc\varphi_0 \quad (23)$$

Also, the expansion of the radial load about $\varphi = \varphi_o$ can be expressed as

$$P(\varphi) = p(\varphi)\epsilon^2 = \sum_{n=2} q^{(n)}(x)\epsilon^n \quad (24)$$

The wide boundary-layer nondimensional stresses and displacements are defined as

$$\tau_{rr} = \frac{\sigma_{rr}^*}{E},\ \tau_{\varphi\varphi} = \frac{\sigma_{\varphi\varphi}^*}{E},\ \tau_{\theta\theta} = \frac{\sigma_{\theta\theta}^*}{E},$$
$$\tau_{r\varphi} = \frac{\sigma_{r\varphi}^*}{E},\ v_r = \frac{u_r^*}{R},\ v_\varphi = \frac{u_\varphi^*}{R} \quad (25)$$

Again, the components of the stresses and displacements are expressed in a power series,

$$g(x,z,\epsilon) = \sum_{n=0}^{\infty} g^{(n)}(x,y)\epsilon^n \quad (26)$$

where $g(x,z,\epsilon)$ is a generic component of the stresses and displacements.

In terms of the wide boundary-layer coordinates, the asymptotic forms of the equilibrium equations are

$$\tau_{r\varphi,z}{}^{(n+1)} = -z\tau_{r\varphi,z}{}^{(n-1)} - 3\tau_{r\varphi}{}^{(n-1)} - \tau_{\varphi\varphi,x}{}^{(n)} + k_1[\tau_{\theta\theta}{}^{(n-1)} - \tau_{\varphi\varphi}{}^{(n-1)}] \quad (27a)$$

$$\tau_{rr,z}{}^{(n+2)} = -z\tau_{rr,z}{}^{(n)} - 2\tau_{rr}{}^{(n)} - \tau_{r\varphi,x}{}^{(n+1)} + [\tau_{\varphi\varphi}{}^{(n)} + \tau_{\theta\theta}{}^{(n)}] - k_1\tau_{r\varphi}{}^{(n)} + xk_2{}^2\tau_{r\varphi}{}^{(n-1)} \quad (27b)$$

while the stress displacement relations may be written as

$$v_{r,z}{}^{(n)} = 0 \quad (28a)$$

$$v_{\varphi,x}{}^{(n+1)} + v_r{}^{(n)} = [\tau_{\varphi\varphi}{}^{(n)} - \nu(\tau_{\theta\theta}{}^{(n)} + \tau_{rr}{}^{(n)})] \quad (28b)$$

$$v_r{}^{(n)} + k_1 v_\varphi{}^{(n)} - xk_2{}^2 v_\varphi{}^{(n-1)} = \tau_{\theta\theta}{}^{(n)} - \nu(\tau_{rr}{}^{(n)} + \tau_{\varphi\varphi}{}^{(n)}) \quad (28c)$$

$$v_{\varphi,z}{}^{(n+1)} + v_{r,x}{}^{(n)} + zv_{\varphi,z}{}^{(n-1)} - v_\varphi{}^{(n-1)} = 2(1+\nu)\tau_{r\varphi}{}^{(n-1)} \quad (28d)$$

It should be noted that the previous sets of equations are only valid for $n = 0,1$ since only terms up to the order ϵ of the cotangent expansion have been used.

The boundary conditions (4) become

$$\tau_{r\varphi}{}^{(n)}(x,\pm 1) = 0;\quad \tau_{rr}{}^{(n)}(x,-1) = 0;$$
$$\tau_{rr}{}^{(n)}(x,+1) = \begin{cases} 0 & \text{, for } n < 2 \\ -q^{(n)}(x), & \text{for } n \geq 2 \end{cases} \quad (29)$$

In addition, matching with the narrow boundary layer suggests that the following integral relations must be imposed (see section on the narrow boundary-layer problem).

$$\int_{-1}^{1} \tau_{\varphi\varphi}(0,z)dz = \int_{-1}^{1} z\tau_{\varphi\varphi}{}^{(n)}(0,z)dz = \int_{-1}^{1} \tau_{r\varphi}{}^{(n)}(0,z)dz = 0 \quad (30)$$

To complete the wide boundary-layer formulation matching with the interior solution must be enforced.

Briefly then, the idea of matching is, for example, that the outer expansion (interior solution) as $\varphi \to \varphi_0$ and the inner expansion (wide boundary layer solution) as $x \to \infty$ are in agreement. If we assume that an overlap region exists, an intermediate limit can be considered. Let

$$\epsilon \to 0;\quad x_\eta = \frac{\varphi - \varphi_0}{\eta(\epsilon)} \quad \text{(fixed)} \quad (31)$$

where

$$\eta/\epsilon \to \infty;\quad \eta \to 0 \quad (32)$$

Note that

$$\varphi - \varphi_0 = \eta x_\eta \to 0;\quad x = \frac{\eta}{\epsilon} x_\eta \to \infty \quad (33)$$

Matching to any order n is thus obtained as follows:

$$\lim_{\substack{\epsilon \to 0 \\ x_\eta \text{ fixed}}} \{f^{(0)} + f^{(1)}\epsilon + \ldots + f^{(n)}\epsilon^n - g^{(0)} - g^{(1)}\epsilon - \ldots g^{(n)}\epsilon^n\} = 0 \quad (34)$$

B. Solution

Substitution of $n = -1, -2$ into the equilibrium equations and the boundary conditions yields

$$\tau_{r\varphi}{}^{(0)} = \tau_{rr}{}^{(0)} = \tau_{rr}{}^{(1)} \quad (35)$$

In addition, by observing Eqs. (28a-28b) and invoking the matching condition we also have that

$$v_\varphi{}^{(0)} = U_\varphi{}^{(0)}(\varphi_0) = \bar{V} = \text{const.} \quad (36)$$

Following the same procedure as was used in the interior solution, we obtain the following forms of the displacement functions, stress displacement relations, equilibrium equations and boundary conditions for $n = 0, 1$:

n = 0

Displacements

$$v_r{}^{(0)}(x,z) = V_r{}^{(0)}(x);\quad v_\varphi{}^{(1)}(x,z) = V_\varphi{}^{(1)}(x) - z\frac{dV_r{}^{(0)}}{dx} \quad (37)$$

Stress displacement relations

$$\tau_{\varphi\varphi}{}^{(0)}(x,z) = T_{\varphi\varphi}{}^{(0)}(x) + z\bar{T}_{\varphi\varphi}{}^{(0)}(x);$$
$$\tau_{\theta\theta}{}^{(0)}(x,z) = T_{\theta\theta}{}^{(0)}(x) + z\bar{T}_{\theta\theta}{}^{(0)}(x) \quad (38)$$

where

$$T_{\varphi\varphi}{}^{(0)} = \frac{1}{1-\nu^2}\left[(1+\nu)V^{(0)} + \frac{dV_\varphi{}^{(1)}}{dx} + \nu\cot\varphi_0\bar{V}\right];$$
$$\bar{T}_{\varphi\varphi}{}^{(0)} = \frac{-1}{1-\nu^2}\frac{d^2V_r{}^{(0)}}{dx^2}$$
$$T_{\theta\theta}{}^{(0)} = \frac{1}{1-\nu^2}\left[(1+\nu)V_r{}^{(0)} + \nu\frac{dV_\varphi{}^{(1)}}{dx} + \cot\varphi_0\bar{V}\right];$$
$$\bar{T}_{\theta\theta}{}^{(0)} = \frac{-\nu}{1-\nu^2}\frac{d^2V_r{}^{(0)}}{dx^2} \quad (39)$$

Shearing and radial normal stresses

$$\tau_{r\varphi}{}^{(1)}(x,z) = \frac{1}{2}(1-z^2)\frac{d\bar{T}_{\varphi\varphi}{}^{(0)}}{dx}$$
$$\tau_{rr}{}^{(2)}(x,z) = z/2(z^2-1)[T_{\varphi\varphi}{}^{(0)} + T_{\theta\theta}{}^{(0)} + q^{(2)}/2] + \tfrac{1}{2}(z^2-1)[\bar{T}_{\varphi\varphi}{}^{(0)} + \bar{T}_{\theta\theta}{}^{(0)}] - \tfrac{1}{2}q^{(2)}(1+z) \quad (40)$$

Equilibrium equations

$$\frac{dT_{\varphi\varphi}{}^{(0)}}{dx} = 0;\quad \frac{d^2\bar{T}_{\varphi\varphi}{}^{(0)}}{dx^2} - 3(T_{\varphi\varphi}{}^{(0)} + T_{\theta\theta}{}^{(0)}) = \frac{3}{2}q^{(2)} \quad (41)$$

or in terms of the displacements,

$$(1+\nu)\frac{dV_r{}^{(0)}}{dx} + \frac{d^2V_\varphi{}^{(1)}}{dx^2} = 0$$
$$\frac{d^4V_r{}^{(0)}}{dx^4} + 3(1+\nu)\left[2V_r{}^{(0)} + \frac{dV_\varphi{}^{(1)}}{dx}\right] = \frac{-3}{2}(1-\nu^2)q^{(2)} - 3(1+\nu)\cot\varphi_0\bar{V} \quad (42)$$

The boundary conditions are

$$T_{\varphi\varphi}^{(0)}(0) = \bar{T}_{\varphi\varphi}^{(0)}(0) = \frac{d\bar{T}_{\varphi\varphi}^{(0)}}{dx}(0) = 0;$$
$$[\text{i.e., } \tau_{\varphi\varphi}^{(0)}(0,z) = \tau_{r\varphi}^{(1)}(0,z) = 0] \tag{43}$$

Additional boundary conditions are required and are obtained by matching with the interior solution, i.e.,

$$\lim_{x\to\infty} V_\varphi^{(1)} = U_\varphi^{(1)}(\varphi) + x\,\frac{dU_\varphi^{(0)}}{d\varphi}(\varphi_0),$$
$$\lim_{x\to\infty} T_{\varphi\varphi}^{(0)} = \Sigma_{\varphi\varphi}^{(0)}(\varphi_0) \tag{44}$$
$$\lim_{x\to\infty} \bar{T}_{\varphi\varphi}^{(0)} = 0; \quad \lim_{x\to\infty} \frac{d\bar{T}_{\varphi\varphi}^{(0)}}{dx} = 0$$

where the limits are taken in the asymptotic sense. These conditions represent the matching of the meridional displacement and the stresses on the meridional face. It is readily verified that these conditions are sufficient to insure the matching of all stresses and displacements.

It should be noted that only six of the seven boundary conditions are required for the wide boundary-layer analysis. The extra matching condition [the second of Eqs. (44)] is needed to complete the interior problem. The first equilibrium equation of Eq. (41) and the first boundary condition of Eq. (43) yield $T_{\varphi\varphi}^{(0)} = 0$. Then from the second matching condition of (44), $\Sigma_{\varphi\varphi}^{(0)}(\varphi_0) = 0$. Thus, the zeroth order interior solution satisfies all boundary conditions identically, and therefore the zeroth order wide boundary-layer solution is nothing more than the asymptotic expansion of the interior solution into this region, i.e., $g^{(0)} = f^{(0)}(\varphi_0)$. The results for the next term are:

n = 1

Displacements

$$v_r^{(1)}(x,z) = V_r^{(1)}(x);$$
$$v_\varphi^{(2)}(x,z) = V_\varphi^{(2)}(x) + z\left(\bar{V} - \frac{dV_r^{(1)}}{dx}\right) \tag{45}$$

Stress displacement relations

$$\tau_{\varphi\varphi}^{(1)}(x,z) = T_{\varphi\varphi}^{(1)}(x) + z\bar{T}_{\varphi\varphi}^{(1)}(x);$$
$$\tau_{\theta\theta}^{(1)}(x,z) = T_{\theta\theta}^{(1)}(x) + z\bar{T}_{\theta\theta}^{(1)}(x) \tag{46}$$

where

$$T_{\varphi\varphi}^{(1)} = \frac{1}{1-\nu^2}\left[(1+\nu)V_r^{(1)} + \frac{dV_\varphi^{(2)}}{dx} + \nu\cot\varphi_0 V_\varphi^{(1)} - \nu csc^2\varphi_0 x\bar{V}\right]$$
$$\bar{T}_{\varphi\varphi}^{(1)} = \frac{-1}{1-\nu^2}\left[\frac{d^2V_r^{(1)}}{dx^2} + \nu\cot\varphi_0\frac{dV_r^{(0)}}{dx}\right]$$
$$T_{\theta\theta}^{(1)} = \frac{1}{1-\nu^2}\left[(1+\nu)V_r^{(1)} + \nu\frac{dV_\varphi^{(2)}}{dx} + \cot\varphi_0 V_\varphi^{(1)} - csc^2\varphi_0 x\bar{V}\right] \tag{47}$$
$$\bar{T}_{\theta\theta}^{(1)} = \frac{-1}{1-\nu^2}\left[\nu\frac{d^2V_r^{(1)}}{dx^2} + \cot\varphi_0\frac{dV_r^{(0)}}{dx}\right]$$

Shearing and radial normal stresses

$$\tau_{r\varphi}^{(2)}(x,z) = \frac{1}{2}(1-z^2)\left[\frac{d\bar{T}_{\varphi\varphi}^{(1)}}{dx} + \cot\varphi_0(\bar{T}_{\varphi\varphi}^{(0)} - \bar{T}_{\theta\theta}^{(0)})\right]$$
$$\tau_{rr}^{(3)}(x,z) = \tfrac{1}{2}z(z^2-1)[T_{\varphi\varphi}^{(1)} + T_{\theta\theta}^{(1)} + q^{(3)}/2] + \tfrac{1}{2}(z^2-1)[\bar{T}_{\varphi\varphi}^{(1)} + \bar{T}_{\theta\theta}^{(1)}] - \tfrac{1}{2}q^{(3)}(1+z) \tag{48}$$

Equilibrium equations

$$\frac{dT_{\varphi\varphi}^{(1)}}{dx} = \cot\varphi_0(T_{\theta\theta}^{(0)} - T_{\varphi\varphi}^{(0)})$$
$$\frac{d^2\bar{T}_{\varphi\varphi}^{(1)}}{dx^2} - 3[T_{\varphi\varphi}^{(1)} + T_{\theta\theta}^{(1)}] = \frac{3}{2}q^{(3)} + \cot\varphi_0\left(\frac{d\bar{T}_{\theta\theta}^{(0)}}{dx} - 2\frac{d\bar{T}_{\varphi\varphi}^{(0)}}{dx}\right) \tag{49}$$

or, in terms of displacements,

$$(1+\nu)\frac{dV_r^{(1)}}{dx} + \frac{d^2V_\varphi^{(2)}}{dx^2} = (\nu + \cot^2\varphi_0)\bar{V} - \frac{dV_\varphi^{(1)}}{dx}$$
$$\frac{d^4V_r^{(1)}}{dx^4} + 3(1+\nu)\left[2V_r^{(1)} + \frac{dV_\varphi^{(2)}}{dx}\right] = -\frac{3}{2}(1-\nu^2)q^{(3)} - (1+\nu)[\cot\varphi_0 V_\varphi^{(1)} - csc^2\varphi_0 x\bar{V}] + (1-\nu^2)\cot\varphi_0\left[\frac{d\bar{T}_{\theta\theta}^{(0)}}{dx} - 2\frac{d\bar{T}_{\varphi\varphi}^{(0)}}{dx}\right] \tag{50}$$

The boundary conditions are

$$T_{\varphi\varphi}^{(1)}(0) = \bar{T}_{\varphi\varphi}^{(1)}(0) = 0, \frac{d\bar{T}_{\varphi\varphi}^{(1)}}{dx}(0) = \cot\varphi_0\bar{T}_{\theta\theta}^{(0)}(0)$$
$$[\text{i.e., } \tau_{\varphi\varphi}^{(1)}(0,z) = \tau_{r\varphi}^{(2)}(0,z) = 0] \tag{51}$$

and are supplemented with the following matching conditions

$$\lim_{x\to\infty} V_\varphi^{(2)} = U_\varphi^{(2)}(\varphi_0) + x\frac{dU_\varphi^{(1)}}{d\varphi}(\varphi_0) + \frac{1}{2}x^2\frac{d^2U_\varphi^{(0)}}{d\varphi^2}(\varphi_0)$$
$$\lim_{x\to\infty} T_{\varphi\varphi}^{(1)} = \Sigma_{\varphi\varphi}^{(1)}(\varphi_0) + x\frac{d\Sigma_{\varphi\varphi}^{(0)}}{d\varphi}(\varphi_0) \tag{52}$$
$$\lim_{x\to\infty} \frac{d\bar{T}_{\varphi\varphi}^{(1)}}{dx} - \cot\varphi_0\bar{T}_{\theta\theta}^{(0)} = 0, \ \lim_{x\to\infty}\bar{T}_{\varphi\varphi}^{(1)} = 0$$

In concluding this section, we wish to add that the equilibrium equations and stress displacement relations for $n = 0,1$ are identical to those obtained by rewriting the Donnell equations in terms of the wide boundary-layer coordinates, and expanding into a power series of ϵ. The shearing and radial normal stresses [Eqs. (40) and (48)] are not contained in the Donnell theory.

The Narrow Boundary-Layer Problem

A. Formulation

From St. Venant's principle it would be expected that the width of the narrow boundary layer is of the order of the shell's thickness, and we therefore introduce the stretched coordinate

$$y = (\varphi - \varphi_0)/\epsilon^2 \tag{53}$$

As was the case for the wide boundary layer, this transformation introduces ϵ into the cotangent function, whose Taylor series expansion about φ_0 may be written as

$$\cot\varphi = k_1 + \epsilon^2 k_2{}^2 y + \ldots \tag{54}$$

where k_1 and k_2 have been previously defined, Eq. (23).

The nondimensional narrow boundary-layer stresses and displacements are

$$s_{rr} = \frac{\sigma_{rr}^*}{E},\ s_{\varphi\varphi} = \frac{\sigma_{\varphi\varphi}^*}{E};\quad s_{\theta\theta} = \frac{\sigma_{\theta\theta}^*}{E},\ s_{r\varphi} = \frac{\sigma_{r\varphi}^*}{E};$$
$$w_r = \frac{u_r^*}{R},\ w_\varphi = \frac{u_\varphi^*}{R} \tag{55}$$

Let $h(y,x,\epsilon)$ denote any generic component of the stresses or displacements, and again assume that they can be expressed as a power series,

$$h(y,z,\epsilon) = \sum_{n=0}^{\infty} h^{(n)}(y,z)\epsilon^n \tag{56}$$

where $h^{(n)} = 0$ for $n < 0$. In addition, expanding the load about $\varphi = \varphi_0$ yields

$$P(\varphi) = \sum_{n=2}^{\infty} \bar{q}^{(n)}(y)\epsilon^n \tag{57}$$

where $\bar{q}^{(2)}(y) = p(\varphi_0)$.

B. Solution

The equilibrium equations can be written as

$$s_{rr,z}^{(n)} + s_{r\varphi,y}^{(n)} = 0;\quad s_{r\varphi,z}^{(n)} + s_{\varphi\varphi,y}^{(n)} = 0 \tag{58}$$

the stress displacement relationships as

$$w_{r,z}^{(n)} = 0;\quad w_{\varphi,y}^{(n)} = 0;\quad w_{\varphi,z}^{(n)} - w_{r,y}^{(n)} = 0$$
$$w_r^{(n)} + k_1 w_\varphi^{(n)} + \nu(s_{rr}^{(n)} + s_{\varphi\varphi}^{(n)}) = s_{\theta\theta}^{(n)} \tag{59a–d}$$

and the compatibility equations as

$$s_{\theta\theta,yz}^{(n)} - \frac{\nu}{1+\nu} s_{,yz}^{(n)} = 0$$
$$s_{rr,yy}^{(n)} - 2s_{r\varphi,yz}^{(n)} + s_{\varphi\varphi,zz}^{(n)} - \frac{\nu}{1+\nu} \times [s_{,yy}^{(n)} + s_{,zz}^{(n)}] = 0 \tag{60a–d}$$
$$s_{\theta\theta,yy}^{(n)} - \frac{\nu}{1+\nu} s_{,yy}^{(n)} = 0;\quad s_{\theta\theta,zz}^{(n)} - \frac{\nu}{1+\nu} s_{,zz}^{(n)} = 0$$

where

$$s^{(n)} = s_{rr}^{(n)} + s_{\varphi\varphi}^{(n)} + s_{\theta\theta}^{(n)} \tag{61}$$

The previous equations are only valid for $n = 0,1$ because terms of higher order than ϵ have been neglected in the expansion of $\cot\varphi$ [Eq. (54)].

The boundary conditions are

$$s_{r\varphi}^{(n)}(y,\pm1) = s_{rr}^{(n)}(y,-1) = 0;\quad s_{rr}^{(n)}(y,+1) = 0,\ n < 2$$
$$s_{rr}^{(n)}(y,+1) = -\bar{q}^{(n)}(y),\ n \geq 2; \tag{62}$$
$$s_{\varphi\varphi}^{(n)}(0,z) = s_{r\varphi}^{(n)} = 0$$

A stress function $\psi^{(n)}(y,z)$ is defined as

$$\psi_{,yy}^{(n)} = s_{rr}^{(n)} - (\tau_{rr})_A{}^{(n)};$$
$$\psi_{,zz}^{(n)} \equiv s_{\varphi\varphi}^{(n)} - (\tau_{\varphi\varphi})_A{}^{(n)}; \tag{63}$$
$$\psi_{,yz}^{(n)} \equiv (\tau_{r\varphi})_A{}^{(n)} - s_{r\varphi}^{(n)}$$

where $(\tau_{ij})_A{}^{(n)}$ is the asymptotic expansion of the wide boundary-layer stress into the narrow boundary layer, and may be expressed as

$$(\tau_{ij})_A{}^{(n)} = \sum_{m=0}^{n} \frac{1}{m!}\frac{d^m\tau_{ij}^{(n-m)}}{dx^m}(0,z)y^m \tag{64}$$

Substituting the above into the last of the stress displacement relations results in the following expression for the circumferential component of stress

$$s_{\theta\theta}^{(n)} = \nu\nabla^2\psi^{(n)} + \tau_{\theta\theta A}{}^{(n)} \tag{65}$$

The equilibrium equations (58) and all but the second of the compatibility equations (60) are identically satisfied when use is made of the definitions of Eqs. (63) and the relationship of Eq. (65). The compatibility equation (60b) reduces to

$$\nabla^4\psi^{(n)} = 0 \tag{66}$$

where

$$\nabla^4(\ ) = (\ )_{,yyyy} + 2(\ )_{,yyzz} + (\ )_{,zzzz}$$

is the biharmonic operator in cartesian coordinates. It should be noted that in general Eqs. (62–65) are only valid in these simple forms for $n \leq 3$.[25]

Boundary conditions (62) reduce to the following conditions on the stress function

$$\psi_{,yy}^{(n)}(y,\pm1) = \psi_{,yz}^{(n)}(y,\pm1) = 0 \tag{67}$$

and

$$\psi_{,zz}^{(n)}(0,z) = -\tau_{\varphi\varphi}^{(n)}(0,z);\quad \psi_{,yz}^{(n)}(0,z) = \tau_{r\varphi}^{(n)}(0,z) \tag{68}$$

Finally, matching results in

$$\lim_{y\to\infty}\psi_{,zz}^{(n)}(y,z) = 0;\quad \lim_{y\to\infty}\psi_{,yz}^{(n)}(y,z) = 0 \tag{69}$$

The displacements are obtained by solving Eqs. (59a–c) and determining the integration unknowns by matching.

Summarizing, we may write the stress function formulation for $n \leq 3$ as follows[25]

$$\nabla^4\psi^{(n)} = 0$$
$$\psi_{,zz}^{(n)}(0,z) = -\tau_{\varphi\varphi}^{(n)}(0,z);\quad \psi_{,yz}^{(n)}(0,z) = \tau_{r\varphi}^{(n)}(0,z);$$
$$\psi_{,yy}^{(n)}(y,\pm1) = 0;\quad \psi_{,yz}^{(n)}(y,\pm1) = 0; \tag{70}$$
$$\lim_{y\to\infty}\sigma_{,zz}^{(n)}(y,z) = 0;\quad \lim_{y\to\infty}\psi_{,yz}^{(n)}(y,z) = 0$$

C. Establishment of the Integral Relations

In order to establish the integral relations a mathematical theorem due to Reiss[16] is stated. Consider a function $\psi(y,z)$ defined on a semi-infinite strip $y \geq 0$, $|z| \leq 1$, which satisfies the partial differential equation

$$\nabla^4\psi = 0 \tag{71a}$$

and having boundary conditions

$$\psi_{,zz}(0,z) = Q(z),\ \psi_{,yz}(0,z) = L(z) \tag{71b}$$
$$\psi_{,yy}(y,\pm1) = 0,\ \psi_{,yz}(y,\pm1) = 0 \tag{71c}$$
$$\lim_{y\to\infty}\psi_{,zz}(y,z) = 0,\ \lim_{y\to\infty}\psi_{,yz}(y,z) = 0 \tag{71d}$$

and $\psi_{,zz}$ $\psi_{,yz}$ and $\psi_{,z}$ are uniformly continuous functions of z as $y \to \infty$, then for ψ, $\psi_{,y}$ and $\psi_{,z}$ to be single valued functions

$$\int_{-1}^{1} L(z)dz = \int_{-1}^{1} Q(z)dz = \int_{-1}^{1} zQ(z)dz = 0 \tag{72}$$

Application of this theorem to the narrow boundary-layer formulation (70) yields the integral relationships

$$\int_{-1}^{1}\tau_{\varphi\varphi}^{(n)}(0,z)dz = 0,\ \int_{-1}^{1} z\tau_{\varphi\varphi}^{(n)}(0,z)dz = 0,$$
$$\int_{-1}^{1}\tau_{r\varphi}^{(n)}(0,z)dz = 0 \tag{73}$$

which are the boundary conditions for the wide boundary-layer stresses. The narrow boundary-layer formulation (70)

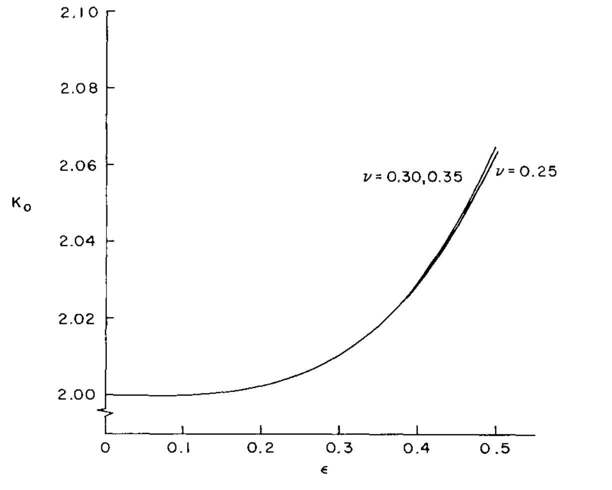

Fig. 2 Stress concentration factor at median surface ($\varphi = \varphi_0 = 0.1$).

may then be viewed as a plain strain problem of a semi-infinite strip subjected to end loads. The integral relations (73) are an expression of the fact that these loads are self-equilibrating.

D. General Solution to the Biharmonic Problem

The general solution to the biharmonic problem (71) can be expressed in terms of the Fadle-Papkovich functions[18,19]

$$\psi_{2k} = e^{-\alpha_{2k}y}[\cos\alpha_{2k}z - z\cot\alpha_{2k}\sin\alpha_{2k}z];\quad k = 1,2,3,\ldots \tag{74a}$$

and

$$\psi_{2k+1} = e^{-\alpha_{2k+1}y}[\sin\alpha_{2k+1}z - z\tan\alpha_{2k+1}\cos\alpha_{2k+1}z] \tag{74b}$$

where the eigenvalues α are chosen so as to satisfy the boundary conditions at $z = \pm 1$ (71c), and are therefore the solutions of the transcendental equations

$$\sin 2\alpha_{2k} + 2\alpha_{2k} = 0 \quad \text{and} \quad \sin 2\alpha_{2k+1} - 2\alpha_{2k+1} = 0 \tag{75}$$

Equation (74a) yields solutions which are even in z, while Eq. (74b) yields solutions odd in z. If α is a solution of Eqs. (75) then its conjugate α^* is a solution, and so is $-\alpha$ and $-\alpha^*$. Of the four, α is chosen to have both positive real and imaginary parts, then $-\alpha$ and $-\alpha^*$ must be discarded to fit the matching conditions (71d).

From Eqs. (74) it follows that

$$\psi_q(y,z,\alpha^*) = \psi_q^*(y,z,\alpha)$$

from which the general solution, written in terms of the real and imaginary parts, follows

$$\psi = \sum_{q=2}^{\infty} \{C_q[\psi_q] + D_q[\psi_q]\} \tag{76}$$

where C_q and D_q are constants determined from the boundary conditions (71b).

E. Numerical Calculations

a) Eigenvalues

A method applicable to determining the eigenvalues was proposed by T. R. Kane.[26] If α is assumed to be a function of the parameter t, $0 \leq t \leq 1$, satisfying

$$F(\alpha) = F(\zeta)(1 - t) \tag{77}$$

then $\alpha(1)$ is a solution to the eigenvalue problem $F(\alpha) = 0$, and ζ is the value of α when $t = 0$. Equation (77) is the solution of the differential equation

$$(dF/d\alpha)d\alpha/dt = -F(\zeta) \tag{78}$$

with the initial condition $\alpha(0) = \zeta$. The eigenvalues are determined by numerically integrating Eq. (78) with a properly chosen initial condition. Utilizing the above differential equation, where F is given by Eq. (75), and separating real and imaginary parts, results in

$$d\rho_{2k}/dt = (M_1M_2 - M_3M_4)/D_1,\qquad d\gamma_{2k}/dt = -(M_1M_4 + M_2M_3)/D_1 \tag{79a}$$

and

$$d\rho_{2k+1}/dt = (M_5M_6 - M_7M_8)/D_2,\qquad d\gamma_{2k+1}/dt = (M_5M_8 + M_6M_7)/D_2 \tag{79b}$$

where

$$\alpha_q = \rho_q + i\gamma_q,\ \zeta_q = l_q + im_q$$

$$M_1 = (\cos 2l_{2k}\sinh 2m_{2k} + 2m_{2k}),\ M_2 = \sin 2\rho_{2k}\sinh 2\gamma_{2k}$$

$$M_3 = (\sin 2l_{2k}\cosh 2m_{2k} + 2l_{2k}),\ M_4 = (\cos 2\rho_{2k}\cosh 2\gamma_{2k} + 1)$$

$$M_5 = (2l_{2k+1} - \sin 2l_{2k+1}\cosh 2m_{2k+1}),$$

$$M_6 = (\cos 2\rho_{2k+1}\cosh 2\gamma_{2k+1} - 1) \tag{80}$$

$$M_7 = (2m_{2k+1} - \cos 2l_{2k+1}\sinh 2\gamma_{2k+1}),$$

$$M_8 = \sin 2\rho_{2k+1}\sinh 2\gamma_{2k+1}$$

$$D_1 = 2M_2^2 + 2M_4^2,\ D_2 = 2M_6^2 + 2M_8^2$$

which were integrated by using a Runge-Kutta scheme. The first several values of α were previously determined.[27,28] To determine additional values, the initial conditions used in each case were obtained by increasing the previously determined eigenvalue by an increment which was less than the difference of the previous two values (simply the first significant digit of the difference). The increment of the parameter

Table 1 First 30 roots of transcendental equation (V.23)

Index q	Root α_q
2	2.106196 + i 1.125364
3	3.748838 + i 1.384339
4	5.356269 + i 1.551574
5	6.949980 + i 1.676105
6	8.536682 + i 1.775544
7	10.11926 + i 1.858384
8	11.69918 + i 1.929405
9	13.27727 + i 1.991571
10	14.85406 + i 2.046852
11	16.42987 + i 2.096626
12	18.00493 + i 2.141891
13	19.57941 + i 2.183398
14	21.15341 + i 2.221723
15	22.72704 + i 2.257320
16	24.30034 + i 2.290552
17	25.87338 + i 2.321714
18	27.44620 + i 2.351048
19	29.01883 + i 2.378758
20	30.59130 + i 2.405013
21	32.16362 + i 2.429958
22	33.73581 + i 2.453719
23	35.30790 + i 2.476403
24	36.87989 + i 2.498102
25	38.45180 + i 2.518900
26	40.02363 + i 2.538867
27	41.59539 + i 2.558068
28	43.16709 + i 2.576559
29	44.73873 + i 2.594391
30	46.31032 + i 2.611609
31	47.88187 + i 2.628254

t was taken to be 0.002, which was halved whenever necessary in order to retain seven significant figure accuracy. Calculations were performed on the CDC 6600 computer located at the Courant Institute of Mathematical Sciences of New York University, and the results for the first thirty eigenvalues are tabulated in Table 1.

b) Least-square point matching

A number of methods suitable for determining the coefficients C_q and D_q have been proposed. They are: expanding of the nonorthogonal series into an orthogonal one; minimizing the integral of the square error; point matching; and least-square point matching. Of these, the least-square point matching technique is the most efficient for use in the present problem.[20] It has also been pointed out that the least-square point matching technique when applied to harmonic and biharmonic problems (as is the case here) leads to numerical errors which are maximum on the boundary.[22] Therefore, we can establish an upper bound for the errors, and thereby, reduce them to any desired level by using more terms in the series.

Essentially, the least-square point matching technique consists of choosing more boundary points for matching than the number of terms included in the truncated series, which therefore results in an over-determined system. The coefficients are then found by minimizing the sum of the squares of the error involved at each point. Nudenfuhr, Leissa, and Lo[22] have found that best results are obtained when the system is doubly determined, that is when the number of matching points are twice the number of terms of the series. They have also shown that if the over determined system of linear equations is written in matrix form:

$$AX = R$$

where A is an $i \times j$ matrix, $i > j$, formed by evaluating the Fadle-Papkovich functions at the boundary, X is the j column matrix of the unknown coefficients, and R is the i column matrix whose elements are the boundary values at the matched points, then the least square procedure produces an $j \times j$ system of equations of the form

$$A^T A X = A^T R \qquad (81)$$

where A^T is the transpose of the matrix A.

F. Solutions

It can be easily shown that the zero-th and first-order displacements and stresses are equivalent to the asymptotic expansion of the wide boundary-layer functions into the narrow boundary layer.

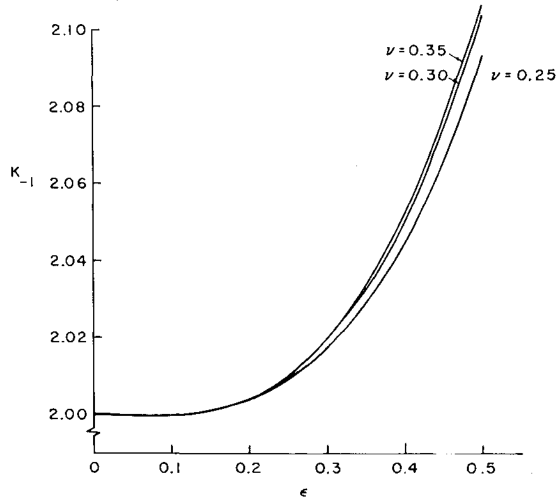

Fig. 3 Stress concentration factor at inner surface ($\varphi = \varphi_0 = 0.1$).

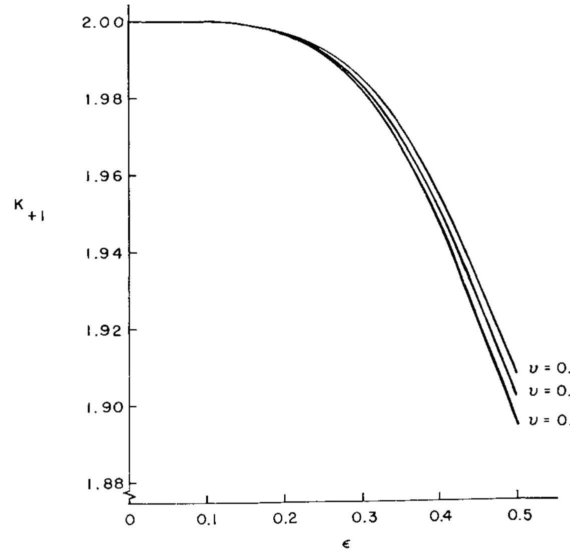

Fig. 4 Stress concentration factor at outer surface ($\varphi = \varphi_0 = 0.1$).

Since we have shown that the zero-th and first-order interior solutions satisfy the given boundary conditions (5) identically, the zero-th and first order stresses in the narrow boundary layer are also the asymptotic expressions of the interior stresses into the narrow boundary layer. Thus the second- and third-order displacements (which are only functions of the zero-th and first-order stresses) in the narrow boundary layer are also the asymptotic expansion into the narrow boundary of the wide boundary-layer displacements. The second- and third-order stresses for the boundary layer are determined by the procedure previously stated.

Uniformly Valid Solution

Let F represent any stress or displacement which is uniformly valid. To obtain a uniformly valid solution the corresponding solutions for all three regions are summed and their common parts subtracted out. These common parts are the portions of the regional solutions which are valid in an overlap region. For the overlap between the interior and wide boundary layer the common part is the asymptotic expansion of the interior into the wide boundary layer. For the overlap between the wide and narrow boundary layers it is the asymptotic expansion of the wide boundary layer into the narrow boundary layer. Therefore, defining $F^{(N)}$ as a uniformly valid thick shell theory of order N, we have

$$F^{(N)} = \sum_{n=0}^{N} \left[f^{(n)}(\varphi,z) + g^{(n)}\left(\frac{\varphi - \varphi_0}{\epsilon}, z\right) + h^{(n)}\left(\frac{\varphi - \varphi_0}{\epsilon^2}, z\right) - (f)_A{}^{(n)} - (g)_A{}^{(n)} \right] \epsilon^n \qquad (82)$$

where $(f)_A{}^{(n)}$ and $(g)_A{}^{(n)}$ are functions of φ and z, and are the asymptotic expansions of the interior solutions into the wide boundary layer, and the wide boundary-layer solutions into the narrow boundary layer, respectively.

Discussion of Results

It has been shown that the equilibrium equations and the stress displacement relations of the interior problem for $n = 0,1$ are identical to those of the classical membrane theory. The radial stress component, of course, is not contained in the membrane theory. It has also been shown that in the wide boundary layer, the equilibrium equations and stress displacement relations for $n = 0,1$ are identical to those ob-

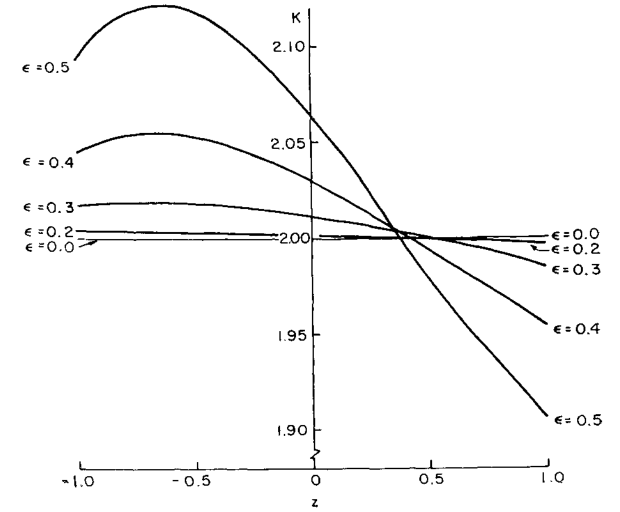

Fig. 5 Thickness variation of K at edge of hole ($\nu = \frac{1}{4}$).

tained by rewriting the Donnell equations in terms of the wide boundary layer coordinates, and then expanding into a power series in ϵ. The shearing and radial normal stresses which are also obtained are, however, not contained in the Donnell Theory.

A fourth-order uniformly valid solution for the case of constant external pressure (p) acting on a spherical shell containing two diametrically opposite holes is presented in Appendix A. The use of these forms for the case when each of the holes is subtended by a half angle of one-tenth of a radian (i.e., $\varphi_o = 0.1$), have led to the numerical results which are presented in Figs. 2–11.

The stress concentration factor K which will be referred to, is defined as the ratio of the circumferential stress at the edge of the hole to that which would be obtained from the elasticity solution for a complete shell.

The variation, at the hole, of the median surface stress concentration factor K_o with ϵ is shown in Fig. 2 for several values of Poisson's ratio. For $\epsilon = 0$, $K = 2.0$, which coincides with the plane stress results for an infinite plate with a circular hole. This quantity increases with increasing, but small, values of ϵ, and, for example for $\epsilon = 0.3$, the percentage increase is only about 0.5%. Similar results are presented in Figs. 3 and 4 for the stress concentration factors in the inner (K_{-1}) and outer (K_{+1}) shell surfaces. Since the first term containing Poisson's ratio is of fourth-order (ϵ^4), only minor variations with ν are observed. It is also seen that the stress concentration is larger at the inner surface than at either of the other two surfaces. The variation through the thickness

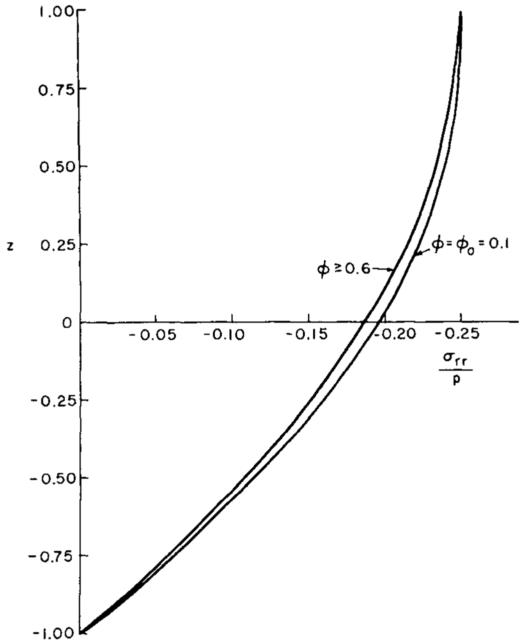

Fig. 7 Normal radial stress ($\varphi_0 = 0.1$, $\epsilon = 0.5$).

of the concentration factor at the edge of the hole with respect to z is shown in Fig. 5 for varying ϵ and $\nu = \frac{1}{4}$. The factor is greatest in the inner region of the shell and attains its maximum value somewhere in between the inner and median surfaces.

Figure 6 shows the variation with φ of the median surface stress concentration factor for $\nu = \frac{1}{4}$. This graph illustrates that the boundary-layer effect is relatively insensitive to changes in ϵ. The radial normal stress (which is independent of ν) as a function of z is presented in Fig. 7 for $\epsilon = 0.5$. It is seen that the boundary-layer effect essentially vanishes when $\varphi \geq 0.6$, i.e., the boundary region extends for a distance of $2\epsilon^2$.

In Fig. 8 the y variation of $\sigma_{r\varphi}$ is given for $z = 0.5$. The shearing stress, which is a narrow boundary-layer phenomenon, is independent of ν and essentially vanishes at a distance of $2\epsilon^2$ from the edge of the hole. The variation of the shearing stress through the thickness is illustrated in Fig. 9 at $\varphi = 0.23$, which is in the vicinity of the peak values of these stresses (see Fig. 8).

Figures 10 and 11 show the radial median surface and meridional (which is independent of z) displacements as a function of φ for $\nu = 0.3$ and varying ϵ. It is observed that for $\epsilon < 0.3$ the displacements vary only slightly from their values obtained from membrane theory.

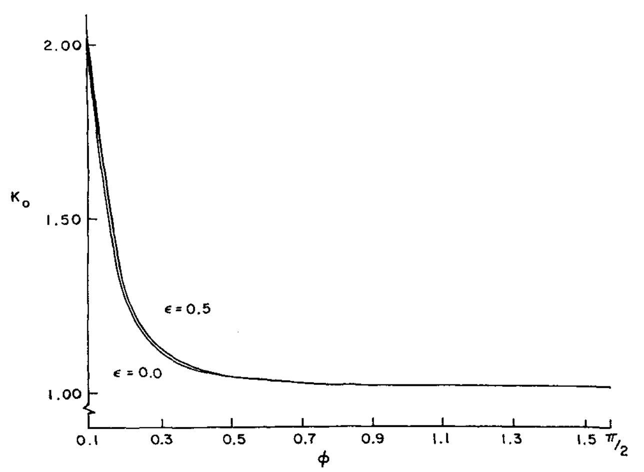

Fig. 6 Variation of K_0 with φ ($\nu = \frac{1}{4}$).

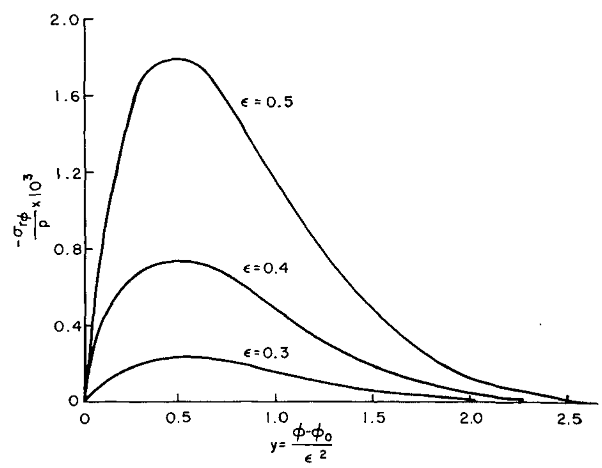

Fig. 8 "Maximum" shearing stress ($z = 0.5$).

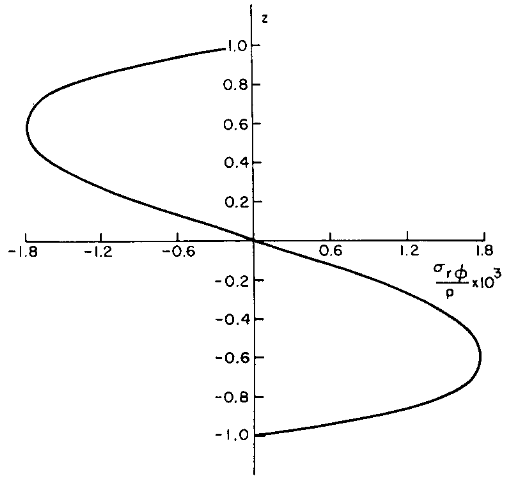

Fig. 9 Shearing stress profile at $\varphi = 0.23$ ($\epsilon = 0.5$).

Concluding Remarks

If the narrow boundary-layer thickness y^*, is defined as that value of y at which the narrow boundary-layer stresses are 5% of their peak values, then from Fig. 8,

$$y^* \simeq 2,$$

so that the extent of the narrow boundary layer is of the order of the shell thickness h. It has been shown that the slowest decaying exponential in the narrow boundary has the form e^{-2y} [see Eq. (74) and Table 1], which leads to the same conclusion as above.

The wide boundary-layer solutions decay as $e^{-\beta x}$ [25] (see Appendix A). Using the same definition for the wide boundary-layer width, and assuming that $\beta \simeq 1$, results in a wide boundary-layer thickness $x^* \simeq 3.0$. Thus, wide boundary-layer effects may be neglected at distances greater than $2(Rh)^{1/2}$ from the hole.

Appendix A: Constant External Pressure

Solutions were found for the special case of a constant external pressure by retaining all terms up to and including the fourth order. The results obtained from an Nth order shell theory (4th-order uniformly valid solution) for the case of a constant external pressure is:[25]

$$\frac{u_\varphi^{(4)}}{p} = \frac{1}{4}(1+\nu)\sin^2\varphi_0\left[\sin\varphi \ln\left(\frac{1-\cos\varphi}{\sin\varphi}\right) - \cot\varphi\right] \times (1 + 2\epsilon^2 + \epsilon^4)$$

$$\frac{u_r^{(4)}}{p} = -\frac{1}{4}(1+\nu)\sin^2\varphi_0\left[1 + \cos\varphi\ln\left(\frac{1-\cos\varphi}{\sin\varphi}\right)\right] \times (1+2\epsilon^2+\epsilon^4) - \frac{1}{4}(1-\nu) - \frac{1}{2}(1-2\nu-\nu z)\epsilon^2 - \frac{1}{6}(1-7\nu)\epsilon^4 - \frac{1}{2}(1-2\nu)z\epsilon^4 - \frac{1}{4}(1+\nu)z^2\epsilon^4 + \frac{\epsilon^4}{4\beta^2}(1-\nu^2)e^{-\beta x}[\sin\beta x - \cos\beta x]$$

$$\frac{\sigma_{\varphi\varphi}^{(4)}}{p} = \frac{1}{4}\left[\frac{\sin^2\varphi_0}{\sin^2\varphi} - 1\right](1+2\epsilon^2 - z\epsilon^2) + \left[\frac{1}{4}\frac{\sin^2\varphi_0}{\sin^2\varphi} - \frac{1}{6}\right]\epsilon^4 - z\left[\frac{1}{2}\frac{\sin^2\varphi_0}{\sin^2\varphi}\right]\epsilon^4 - z^2\left[\frac{1}{2} - \frac{1}{4}\frac{\sin^2\varphi_0}{\sin^2\varphi}\right]\epsilon^4 + \frac{z}{2}e^{-\beta x} \times [\cos\beta x + \sin\beta x]\epsilon^4 + \epsilon^4\bar{\psi}_{,zz}{}^{(4)} \quad \text{(A1)}$$

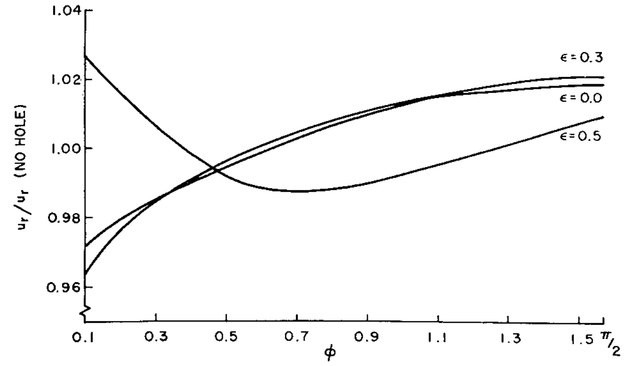

Fig. 10 Radial displacement at median surface ($\nu = 0.3$).

$$\frac{\sigma_{\theta\theta}^{(4)}}{p} = -\frac{1}{4}\left[\frac{\sin^2\varphi_0}{\sin^2\varphi} + 1\right](1+2\epsilon^2 - z\epsilon^2) - \left[\frac{1}{4}\frac{\sin^2\varphi_0}{\sin^2\varphi} + \frac{1}{6}\right]\epsilon^4 + z\left[\frac{1}{2}\frac{\sin^2\varphi_0}{\sin^2\varphi}\right]\epsilon^4 - z^2\left[\frac{1}{2} + \frac{1}{4}\frac{\sin^2\varphi_0}{\sin^2\varphi}\right]\epsilon^4 + \frac{(1-\nu^2)}{4\beta^2}e^{-\beta x} \times [\sin\beta x - \cos\beta x]\epsilon^4 + \frac{\nu}{2}ze^{-\beta x}(\cos\beta x + \sin\beta x)\epsilon^4 + \epsilon^4\nu\nabla^2\bar{\psi}^{(4)}$$

$$\sigma_{r\varphi}{}^{(4)}/p = \epsilon^4\bar{\psi}_{,yz}$$

$$\sigma_{rr}{}^{(4)}/p = (\epsilon^2/2)(1+z) - \epsilon^4(1-z^2) + \epsilon^4\bar{\psi}_{,yy}{}^{(4)} \quad \text{(A2)}$$

where $4\beta^4 = 3(1-\nu^2)$

and $\bar{\psi}^{(4)}$ is the solution of the fourth-order narrow boundary-layer problem

$$\nabla^4\bar{\psi}^{(4)} = 0$$
$$\bar{\psi}_{,zz}{}^{(4)}(0,z) = -\tfrac{1}{12} + \tfrac{1}{4}z^2, \quad \bar{\psi}_{,yz}{}^{(4)}(0,z) = 0$$
$$\bar{\psi}_{,yy}{}^{(4)}(y,\pm 1) = \bar{\psi}_{,yz}{}^{(4)}(y,\pm 1) = 0 \quad \text{(A3)}$$
$$\lim_{y\to\infty}[\bar{\psi}_{,zz}{}^{(4)}(y,z),\ \bar{\psi}_{,yz}{}^{(4)}(y,z)] \to 0$$

where $\bar{\psi}^{(4)} = \psi^{(4)}/p$

The terms contained in the uniformly valid solution arise as follows. Terms containing $e^{-\beta x}$ are obtained from the wide boundary layer; those containing $\bar{\psi}^{(4)}$ are obtained from the narrow boundary layer; and the remaining terms are from the interior solutions. For the special loading considered here all boundary-layer terms less than fourth order vanish. The solution of Eqs. (A3) was obtained by matching 30 terms

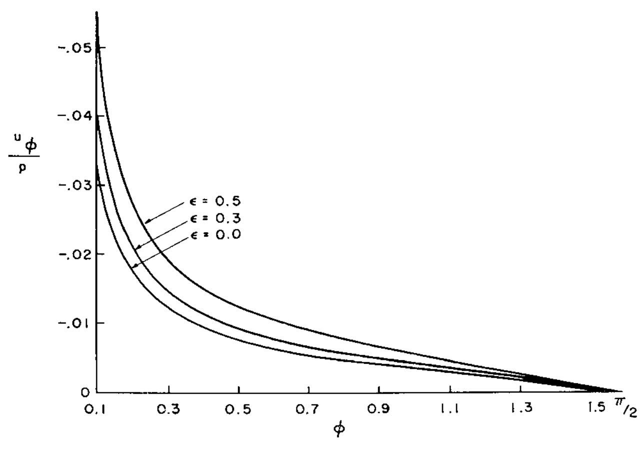

Fig. 11 Meridional displacement ($\nu = 0.3$).

of the Fadle-Papkovitch series (74) at 60 evenly distributed points on the edge $y = 0$, and solving the resulting equations for the unknown coefficients by using Gaussian elimination. The stresses were then checked at 119 points on the boundary (i.e., all points matched and those midway between them) to determine the maximum error involved. In all cases at least four digit accuracy was obtained. Numerical calculations were performed on the IBM 360, Model 50, digital computer located at the Polytechnic Institute of Brooklyn.

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